Lecture 14

Strong Induction, Tips for Good Proofs

When Normal Induction "Fails"

Example: Let $a_0 = 0$, and $a_n = a_0 + a_1 + a_2 + \ldots + a_{n-1} + n$, if $n \geq 1$. Prove that for all non-negative integers n, $a_n = 2^n - 1$.

Solution: We will attempt a proof by (normal) mathematical induction.

Let $P(n)$ be $a_n = 2^n - 1$.

Basis Step: For $n = 0$, $P(0)$ is true because $a_0 = 2^0 - 1$.

Inductive Step: Assume $P(k)$ is true and prove that $P(k+1)$ is true.

$$\begin{array}{ccccccccccccccc} a_{k+1} = & a_0 & + & a_1 & + & a_2 & + & \ldots & + & a_{k-1} & + & a_k & + & k + 1 \\ & \parallel & & \parallel & & \parallel & & & & \parallel & & \parallel & & \\ & 0 & & ? & & ? & & & & ? & & 2^k - 1 & & \end{array}$$

Finding a_{k+1}'s closed form seems difficult without knowing the closed form of a_i's for $i < k + 1$.

As one student pointed out, in the previous example we can simply replace $a_0 + a_1 + \ldots + a_{k-1}$ with $a_k - k$ and complete the proof using normal induction. So, this turned out to be a not so good example where normal induction “fails” and you need stronger induction. But, we will see more examples in tutorials and assignments, where using strong induction makes life easy. Stamps example on the 6th slide is one of them.

Strong Induction

Second Principle of Mathematical Induction:

To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we perform two steps:

Basis Step: Prove that $P(1)$ is true.

Inductive Step: We prove that $(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$ is true for all positive integers k.

How do we perform the inductive step?

Assume $P(1), P(2), \ldots,$ and $P(k)$ are true for any arbitrary positive integer $k,$ and show that under this assumption $P(k+1)$ is also true.

Why Strong Induction Works?

Performing both basis and inductive step, proves that $P(n)$ is true for all positive n because:

- $P(1)$ is true because it was proven in basis step.
- $P(2)$ is true because $P(1)$ and $P(1) \to P(2)$ are true.
- $P(3)$ is true because $P(1)$, $P(2)$, and $(P(1) \wedge P(2)) \to P(3)$ are true.
- $P(4)$ is true because $P(1)$, $P(2)$, $P(3)$, and $(P(1) \wedge P(2) \wedge P(3)) \to P(4)$ are true.
- $\ldots$

Examples: Strong Induction

Let's revisit the previous example.

Example: Let $a_0 = 0$, and $a_n = a_0 + a_1 + a_2 + \ldots + a_{n-1} + n$, if $n \geq 1$. Prove that for all non-negative integers n, $a_n = 2^n - 1$.

Solution: Let $P(n)$ be $a_n = 2^n - 1$.

Basis Step: For $n = 0$, $P(0)$ is true because $a_0 = 2^0 - 1$.

Inductive Step: Assume $P(0), P(1), P(2), \ldots, P(k)$ are true and prove that $P(k+1)$ is true.

$$\begin{aligned} a_{k+1} &= a_0 + a_1 + a_2 + \ldots + a_{k-1} + a_k + k + 1 \\ &= (2^0 - 1) + (2^1 - 1) + \ldots + (2^{k-1} - 1) + (2^k - 1) + k + 1 \\ &= 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k - (k+1) + k + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

$\blacksquare$

Examples: Strong Induction

Example: Prove that every amount of postage of 12 rupees or more can be formed using Rs. 4 and Rs. 5 stamps.

Solution: Let $P(n)$ be the statement that postage of n rupees can be formed using Rs. 4 and Rs. 5 stamps.

Basis Step: $P(12)$, $P(13)$, $P(14)$, $P(15)$ are true because postage of $12, 13, 14, 15$ rupees can be formed with 3 Rs. 4 stamps, 2 Rs. 4 stamps $+$ 1 Rs. 5 stamps, 1 Rs. 4 stamps $+$ 2 Rs. 5 stamps, 3 Rs. 5 stamps.

(Multiple base cases are allowed and helpful sometimes.)

Inductive Step: Assume $P(12), P(13), P(14), \ldots, P(k)$ are true and prove that $P(k+1)$ is true, for any arbitrary $k \geq 15$.

Examples: Strong Induction

$P(k-3)$ is true because $k-3 \geq 12$. That is, we can form postage of $k-3$ rupees using only Rs. 4 and Rs. 5 stamps.

Now, postage of $k+1=(4+k-3)$ rupees can be formed by a Rs. 4 stamps and the stamps we used to form postage of $k-3$ rupees. $\blacksquare$

Tips for Good Proofs

Some tips on writing good proofs. *(Taken from "Maths for CS" by Lehman & Leighton.)*

Read a lot: Read proofs given in solved examples of textbooks.

State your game plan: Whenever possible, explain the general line of reasoning, e.g., "We use strong induction" or "We argue by contradiction", etc.

Explain your reasoning: A good proof usually looks like an essay with some equations thrown in. Do not write long sequence of expressions without explanation.

Tips for Good Proofs

Keep a linear flow: The steps of your argument should follow one another in a clear, sequential order.

Finish: Do not abruptly quit the proof. In the end, tie everything together and explain why the original claim follows.

Simplify: Proof with a fewer logical steps is a better proof than a long, complicated proof.

Don’t be lazy: Use words such as “obviously”, “clearly”, etc., sparingly. Explain why you think something is true.